When $\theta=0$, $d\theta/dt=0$; hence $K=2g$, and we have

$$a\left(\frac{d\theta}{dt}\right)^2=2g(1-\cos\theta)+ac^2\sin^2\theta.$$

Putting $ac^2=ng$, and integrating between limits $\theta=\pi$, $\theta=\frac{1}{2}\pi$, we find

$$t=\sqrt{\frac{a}{(n+1)g}}\log(\sqrt{n+2}+\sqrt{n+1}).$$

(2) The energy of the particle relative to the tube is $\frac{1}{2}m\,a^2\theta'^2$. This is due to two causes, gravity and rotation. Hence

$$\tfrac{1}{2}m\,a^2\theta'^2=amg(1-\cos\theta)+\tfrac{1}{2}m\,a^2\varphi'^2\sin^2\theta,$$

which is the same as the first integral second equation of the Lagrangian groups.

AVERAGE AND PROBABILITY.

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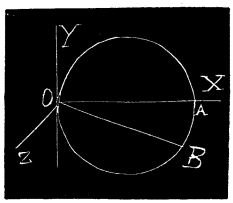
74. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

From a point in the circumference of a circular field a projectile is thrown at random with a given velocity which is such that the diameter of the field is equal to the greatest range of the projectile. Find the chance of its falling into the field. [From Byerly's *Integral Calculus*, page 209.]

I. Solution by the PROPOSER.

It is easily seen that if α, the angle of projection, has a value from $\frac{1}{4}\pi-\frac{1}{2}\phi$ to $\frac{1}{4}\pi+\frac{1}{2}\phi$, the projectile will fall beyond B.

The unfavorable chance is



$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{4}\pi-\frac{1}{2}\phi}^{\frac{1}{4}\pi+\frac{1}{2}\phi}\cos\alpha\,d\alpha\,d\phi=\frac{2}{\pi}(\sqrt{2}-1).$$

Since half of the projectiles will fall on the left of the YZ plane, the favorable chance is $\frac{1}{2}-(2/\pi)(\sqrt{2}-1)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let φ=angle of projection, $OA=2a$, $\angle AOB=\theta$.

$\therefore$ Range$=(v^2/2g)\sin2\varphi=OB=2a\cos\theta$, when the projectile falls within the field.

The range is greatest when $\theta=45°$, and then according to the conditions of the problem $v^2/2g=2a$.

$\therefore$ $2a\sin2\varphi=2a\cos\theta$, or $\sin2\varphi=\cos\theta$.

$\therefore$ $\sin\varphi=\frac{1}{2}(1+\cos\theta)^{\frac{1}{2}}\pm\frac{1}{2}(1-\cos\theta)^{\frac{1}{2}}$.

Therefore the projectile will fall without the circle, if $\sin\varphi$ is less than $\frac{1}{2}(1+\cos\theta)^{\frac{1}{2}}-\frac{1}{2}(1-\cos\theta)^{\frac{1}{2}}$; but will fall within if $\sin\varphi$ is greater than $\frac{1}{2}(1+\cos\theta)^{\frac{1}{2}}+\frac{1}{2}(1-\cos\theta)^{\frac{1}{2}}$.

If all possible directions are equally probable, the chance of the projectile falling within the circle is $1-(1-\cos\theta)^{\frac{1}{2}}=1-\sqrt{2}\sin\frac{1}{2}\theta$.

Hence the required chance is

$$p=\frac{\int_0^{\frac{1}{2}\pi}(1-\sqrt{2}\sin\frac{1}{2}\theta)d\theta}{\int_0^{\pi}d\theta}=\tfrac{1}{2}-(2/\pi)(\sqrt{2}-1).$$

III. **Solution by GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.**

The diameter of the field is v^2/g, and the range must not exceed $(v^2/g)\cos\phi$. The elevation may vary from 0° to θ_1, and from $\frac{1}{2}\pi-\theta_1$ to $\frac{1}{2}\pi$ for each value of ϕ, where θ_1 is determined by $\sin 2\theta_1=\cos\phi$, ϕ the azimuth of the projectile measured from the diameter, θ the elevation of the gun, v the velocity of projection, and g the intensity of gravity.

The surface-element of the enveloping hemisphere whose radius is R or v^2/g is $R^2\cos\theta d\theta d\phi$.

Using only one-half the field and one-fourth of the sphere the required chance is

$$\frac{R^2\int_0^{\frac{1}{2}\pi}\int_0^{\theta_1}\cos\theta d\theta d\phi+R^2\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\theta_1}^{\frac{1}{2}\pi}\cos\theta d\theta d\phi}{\pi R^2}$$

$$\text{or } \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}(1-\sqrt{2}\sin\tfrac{1}{2}\phi)d\phi, \text{ or } \tfrac{1}{2}-\frac{2}{\pi}(\sqrt{2}-1).$$

Solved in a similar manner by *L. C. WALKER*. Solved with different results by *HENRY HEATON*, and *P. H. PHILBRICK*.

75. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find the mean area of all plane rectilineal right triangles having a constant perimeter p.

I. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Let x and y be the base and altitude, respectively.

Then $x+y+\sqrt{(x^2+y^2)}=p$, the perimeter .....(1).

$$\therefore x=\frac{p^2-2py}{2(p-y)}. \quad \therefore \text{Area}=\frac{p}{4}\frac{(p-2y)y}{p-y}.$$